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COMPACT HYPERCOMPLEX AND QUATERNIONIC MANIFOLDS

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1. Introduction

This paper concerns hypercomplex manifolds ([3, §6], [4, pp. 137–139]) and quaternionic manifolds ([3, §1], [4, pp. 135–136]), which are manifolds with a $\mathrm{GL}(n, \mathbb{H})$- and a $\mathrm{GL}(n, \mathbb{H})\mathbb{H}^*$-structure respectively, preserved by a torsion-free connection. It is in two parts, and each part presents a way of constructing compact examples of these manifolds.

In the first part a method is given similar to those used by Gibbons and Hawking to construct hyper-Kähler manifolds and by LeBrun [2] to construct scalar-flat Kähler surfaces. It will be shown that given a hypercomplex or quaternionic manifold M, a Lie group G, an action Ψ of G on M that preserves the structure, and a Ψ-invariant quaternionic G-connection on a principal G-bundle P over M, one can, subject to a certain condition, define a new hypercomplex or quaternionic manifold N that is M "twisted by" the G-bundle P. Here a quaternionic connection is one satisfying a curvature condition that naturally generalizes the instanton equations in the four-dimensional case.

In the second part the theory of homogeneous hypercomplex and quaternionic manifolds will be described. This is based upon the theory of homogeneous complex manifolds given in [5], [8]. The case of homogeneous hypercomplex structures on groups has already been described by Spindel et al. [6].

Both of these methods give many compact, nonsingular, simply-connected hypercomplex and quaternionic manifolds in dimensions greater than four, which are not products or joins of other manifolds, and are not (even locally) hyper-Kähler or quaternionic Kähler. (That is, the structure group cannot be reduced to $\mathrm{SP}(n)$ or $\mathrm{SP}(n)\,\mathrm{SP}(1)$.) We believe that these are the first such examples to be described, other than the homogeneous hypercomplex groups in [6].

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2. A construction for hypercomplex and quaternionic manifolds

Suppose that M is a quaternionic manifold, and let Z be the twistor space of M. Then Z is a complex manifold that is a fiber bundle over M with fibers $\mathbb{CP}^1$ of normal bundle $n\mathcal{O}(1)$, and has an antiholomorphic involution σ that fixes the fibers. Let G be a Lie group, P a principal G-bundle over M, and A a connection on P. The curvature Ω of A is a 2-form on M with values in ad P, where ad P is the bundle associated to the adjoint representation of G on the Lie algebra $\mathfrak{g}$ of G.

At each point of M there is a family of complex structures from the quaternionic structure, and for each such complex structure I, the 2-forms on M can be decomposed into the $+1$- and -1-eigenspaces of I; the $+1$ eigenspace corresponds to the $(1, 1)$-forms and the -1 eigenspace to the $(2, 0)$- and the $(0, 2)$-forms. (Note that we are considering real 2-forms, not complex ones.)

We define a *quaternionic connection* A on P to be a connection whose curvature Ω is in the $+1$-eigenspace for each I in the family at every point. This definition coincides with the definition of a quaternionic connection given in [3, Definition 7.1]. Moreover, when N is four-dimensional, a quaternionic connection is just an (anti-self-dual) instanton. So quaternionic connections are the natural generalization to quaternionic manifolds in all dimensions of the notion of an instanton in four dimensions.

Quaternionic connections are interesting because the following generalization of the Ward correspondence applied to them:

The Ward correspondence. Let M be a quaternionic manifold, Z the twistor space of M, G a Lie group, and P a principal G-bundle over M. Let $\widetilde{P}$ be the lift of P to Z and $\widetilde{P}^c$ the complexification of $\widetilde{P}$, with fiber G^c, the complexification of G. Then quaternionic connections A on P are in one-to-one correspondence with real holomorphic structures on the G^c-bundles $\widetilde{P}^c$ which are trivial on the fibers of Z.

By a real holomorphic structure we mean a holomorphic structure that changes sign under the composition of the real structure of Z and complex conjugation on the fibers of $\widetilde{P}^c$.

Because P is a principal bundle there is an action of G on P, which will be called Φ, that acts transitively on the fibers. Let $\Psi: G \to \mathrm{Aut}(M)$ be an action of G upon M. We choose a lifting of Ψ to P, which will also be called Ψ, preserving the principal bundle structure (i.e., commuting with Φ). This lifting is not necessarily unique up to homotopy.

We shall now prove two theorems, which have very similar statements and proofs.

Theorem 2.1. *Let M, P, Φ, and Ψ be as above, and let A be a Ψ-invariant quaternionic connection on P. Suppose $\Psi(G)$ acts freely on P. Then the manifold $N = P/\Psi(G)$ has a natural (possibly singular) quaternionic structure, which is nonsingular wherever the Lie algebra of $\Psi(G)$ is transverse to the horizontal subspaces of A in P.*

Theorem 2.2. *Let M, P, Φ, and Ψ be as above, and let A be a Ψ-invariant quaternionic connection on P. Let $\Delta: G \to \mathrm{Aut}(P)$ be the diagonal action of G on P, given by $\Delta(g) = \Phi(g)\Psi(g)$. (This is a group homomorphism because Φ and Ψ commute.) Suppose that $\Delta(G)$ acts freely on P. Then the manifold $N = P/\Delta(G)$ has a natural (possibly singular) quaternionic structure, which is nonsingular wherever the Lie algebra of $\Delta(G)$ is transverse to the horizontal subspaces of A in P.*

If M is hypercomplex rather than just quaternionic, and $\Psi(G)$ preserves the hypercomplex structure, then the manifolds N constructed in Theorems 2.1 and 2.2 will also be hypercomplex. This is because if M is hypercomplex then its twistor space Z fibers over $\mathbb{CP}^1$, and this induces a fibration over $\mathbb{CP}^1$ of the twistor space W of N constructed in the proof below. The proofs of Theorems 2.1 and 2.2 are almost identical, so only the first will be given; to get the second, replace Ψ by Δ throughout.

Proof of Theorem 2.1. By the Ward correspondence, the quaternionic connection A on P defines a holomorphic structure on the bundle $\widetilde{P}^c$ over Z. The action Ψ on P lifts to $\widetilde{P}$ and then to $\widetilde{\Psi}$ on $\widetilde{P}^c$, and, as A is Ψ-invariant, this action preserves the holomorphic structure. We define the antiholomorphic involution $\tilde{\sigma}$ of $\widetilde{P}^c$ to be the composition of the antiholomorphic involution σ on Z and complex conjugation on the fibers G^c. Then $\widetilde{\Psi}$ commutes with $\tilde{\sigma}$.

The action $\widetilde{\Psi}$ of G can be complexified to an action $\widetilde{\Psi}^c$ of G^c on $\widetilde{P}^c$. Ideally we would like to say that $\widetilde{P}^c/\widetilde{\Psi}^c(G^c) \cong \widetilde{P}/\widetilde{\Psi}(G)$, because each $\widetilde{\Psi}^c(G^c)$-orbit in $\widetilde{P}^c$ contains exactly one $\widetilde{\Psi}(G)$-orbit in $\widetilde{P}$; thus $\widetilde{P}/\widetilde{\Psi}(G)$ would also be the quotient of a complex manifold by a complex group, and so would have a complex structure. However, this involves us in two sorts of problems: first, some $\widetilde{\Psi}^c(G^c)$-orbits might contain no G-orbits in $\widetilde{P}$, or more than one, and secondly, as G^c is a noncompact group, topological restrictions on its action are necessary for the quotient even to be Hausdorff.

We shall overcome these problems as follows. Let U be a small open neighborhood of G in G^c. We require that U should be invariant under complex conjugation and the action of G on the right, and that the closure of U in G^c should be compact. We also require that U should be sufficiently small such that if $x_1, x_2 \in \widetilde{P}$, $u_1, u_2 \in U$, and $\widetilde{\Psi}^c(u_1)x_1 = \widetilde{\Psi}^c(u_2)x_2$, then x_1, x_2 are the same $\widetilde{\Psi}(G)$-orbit in $\widetilde{P}$. (This is possible at least for compact subsets of $\widetilde{P}$. Here the transversality condition is needed to ensure that the action $\widetilde{\Psi}^c(i\mathfrak{g})$ is transverse to $\widetilde{P}$ in $\widetilde{P}^c$, without which the result might fail.)

Let $S \subset \widetilde{P}^c$ be the set $\widetilde{\Psi}^c(U)[\widetilde{P}]$. Then S is an open neighborhood of $\widetilde{P}$ in $\widetilde{P}^c$, and fibers over $\widetilde{P}/\widetilde{\Psi}(G)$ with fiber U. (Here we use the property of U given in the previous paragraph, and also the right G-invariance of U.) As the fibers are locally $\widetilde{\Psi}^c(G^c)$-orbits, they are complex submanifolds and the fibration is holomorphic. Since $\overline{U}$ is compact, the fibration is topologically well behaved. So $\widetilde{P}/\widetilde{\Psi}(G)$ is the base space of a holomorphic fiber bundle, and is thus a complex manifold. Also, $\tilde{\sigma}$ restricts to S, where it preserves the fibers, so it descends to an antiholomorphic involution σ' of $\widetilde{P}/\widetilde{\Psi}(G)$.

Define $W = \widetilde{P}/\widetilde{\Psi}(G)$. Then, from above, wherever Ψ is transverse to the horizontal subspaces of A, W is the base space of a holomorphic fiber bundle S, and so has a complex structure and an antiholomorphic involution. Moreover, W fibers over $N = P/\Psi(G)$ with fiber $\mathbb{CP}^1$, since dividing by $\Psi(G)$ commutes with passage from M to the twistor space Z. The normal bundle of fibers is $n\mathcal{O}(1)$. This is because the bundle $\widetilde{P}^c$ is trivial over real lines as a holomorphic bundle, and so the normal bundle of a real line in $\widetilde{P}^c$ is $n\mathcal{O}(1) + \mathfrak{g}^c \otimes \mathcal{O}$. But to get the normal bundle of the corresponding fiber of W we have to divide by the part tangent to the orbit of G^c, which is clearly isomorphic to $\mathfrak{g}^c \otimes \mathcal{O}$, leaving $n\mathcal{O}(1)$.

So W is the twistor space for a quaternionic structure on N, which is nonsingular wherever $\Psi(G)$ is transverse to the horizontal subspaces of A. q.e.d.

We note that it is possible to give a rigorous proof of this theorem without invoking the Ward correspondence, by showing that the Nijenhuis tensor of each of the three almost complex structures on the associated bundle of N vanishes, and thus that they are integrable. It is a long but elementary calculation that starts from the fact that the curvature is of type $(1, 1)$ with respect to each complex structure, and has the advantage of avoiding technical problems with complexifying group actions.

2.1. Another construction. We briefly describe a completely different way of using instantons to make hypercomplex and quaternionic manifolds. It is well known that the moduli spaces of instantons on a hyper-Kähler 4-manifold are hyper-Kähler, and that one way of looking at this is to regard the moduli spaces as infinite-dimensional hyper-Kähler quotients of the space of all smooth connections by the gauge group, with the self-duality equations as the moment maps. It is clear that the moduli spaces of instantons on a hypercomplex 4-manifold can be regarded as hypercomplex quotients (in the sense of [1]) in the same way, and thus that the moduli spaces will be hypercomplex manifolds, but not in general compact.

Let X be a hypercomplex Hopf surface. Then $\mathrm{U}(2)$ acts transitively on X permuting the complex structures, in the same way that $\mathbb{H}^*$ acts upon its own complex structures by left multiplication. Let $\mathscr{M}$ be a moduli space of instantons over X; then $\mathscr{M}$ is hypercomplex. The action of $\mathrm{U}(2)$ on X induces an action of $\mathrm{U}(2)$ upon $\mathscr{M}$ that permutes the complex structures of $\mathscr{M}$ in the same way. It is not difficult to see that the quotient of $\mathscr{M}$ by this action of $\mathrm{U}(2)$ will in fact be a quaternionic manifold, where it is nonsingular. We do not know if any of the quaternionic manifolds arising in this fashion can be made compact.

3. Compact hypercomplex and quaternionic manifolds

In this section we will apply Theorem 2.2 to construct compact, nonsingular, simply-connected examples of hypercomplex and quaternionic manifolds. Theorem 2.2 is actually more useful than Theorem 2.1, because there are many situations in which the image of the Lie algebra action ψ is actually contained in the horizontal subspaces of the connection A, and therefore the transversality condition of Theorem 2.1 does not hold anywhere, but that of Theorem 2.2 holds everywhere and so the resulting manifold has a nonsingular quaternionic structure.

Example 1. Let M be a compact simply-connected quaternionic manifold, and suppose that P is a nontrivial, primitive $\mathrm{U}(1)$-bundle (and so, has simply-connected total space) carrying a quaternionic connection A. For instance, M could be $\mathbb{CP}^2$ and the instanton could be the one with curvature form the Kähler form of the Fubini-Study metric; this generalizes to the higher-dimensional symmetric spaces $SU(n+2)/S(U(n) \times U(2))$. Or M could be a self-dual metric on $n\mathbb{CP}^2$ and the instanton that one arising from the harmonic form representing any integral, primitive, nonzero two-dimensional cohomology class.

Now over any quaternionic manifold M there is a bundle $\mathscr{U}(M)$ with fiber $\mathbb{H}^*/\{\pm 1\}$ called the *associated bundle*, which has a hypercomplex structure upon its total space. Salamon defines this bundle in [3, Corollary 7.4], and calls it Y. It is the projectivization of $\mathscr{U}(M)$ with respect to any of the complex structures which is the twistor space Z of M.

The bundle $\mathscr{U}(M)$ is hypercomplex, but not yet compact. Let r be a positive real constant. Then the integers $\mathbb{Z}$ act on $\mathscr{U}(M)$ by multiplication by e^{rn}, $n \in \mathbb{Z}$, and dividing by this action gives a compact hypercomplex manifold $\mathscr{U}(M)/\mathbb{Z}$ which is not simply-connected, and fibers over M with fiber the Hopf surface.

Let Ψ be the action of $U(1)$ on $\mathscr{U}(M)/\mathbb{Z}$ of dilation on the fibers, that is, let $e^{i\theta}$ act by multiplication by $e^{r\theta/2\pi}$. This action preserves the fibration over M and thus the lift of the instanton A to $\mathscr{U}(M)$. Now applying Theorem 2.2 we get a new hypercomplex manifold which is $\mathscr{U}(M)/\mathbb{Z}$ twisted by the nontrivial $U(1)$-bundle P. The new manifold N is compact and simply-connected, because twisting by P kills the fundamental group as P is primitive. It also fibers over M with fiber the Hopf surface, but the $U(1)$-component of the fibration is now nontrivial.

Hopf surfaces will emerge as a recurrent theme in most of the rest of the examples we shall give. An interesting point about the case when M is $\mathbb{CP}^2$ is that the resulting manifold N is homogeneous, that is, has transitive symmetry group. In fact N is $SU(3)$, and has symmetry group $SU(3) \times U(1)$. Homogeneous hypercomplex and quaternionic manifolds will be the topic of the next sections.

A variation on the above is, instead of working with the standard $\mathbb{R}^*_+ \subset \mathbb{H}^*$, to choose a more arbitrary one-parameter subgroup of $\mathbb{H}^*$. Let this act on $\mathscr{U}(M)$ by multiplication, and divide $\mathscr{U}(M)$ by a sublattice of this subgroup. One ends up with a $U(1)$-action upon the quotient of $\mathscr{U}(M)$ by a different action of $\mathbb{Z}$.

The difference in this case is that the actions of $\mathbb{R}$ and $\mathbb{Z}$ do not have to preserve the individual complex structures but only the family, so we are effectively regarding $\mathscr{U}(M)$ as a quaternionic manifold instead of a hypercomplex manifold. The manifold that is then constructed is compact, simply-connected, and quaternionic but not (for general one-parameter subgroups) hypercomplex, and will fiber over M with fiber a Hopf surface, but this time a Hopf surface that is not the quotient of $\mathbb{H}\setminus\{0\}$ by a group of dilations, but by a group generated by left multiplication by a general nonunit quaternion.

Example 2. Let M_1 and M_2 be compact, simply-connected quaternionic manifolds. Now the product of two quaternionic manifolds is not quaternionic, but there is a notion analogous to a product for quaternionic manifolds, called the *join*. The join $M_1 * M_2$ of M_1, M_2 is defined to be the quaternionic manifold with associated bundle $\mathscr{U}(M_1) \times \mathscr{U}(M_2)$. This is an associated bundle because the product of the two hypercomplex manifolds $\mathscr{U}(M_j)$ is hypercomplex, and has an $\mathbb{H}^*$-action given by combining the $\mathbb{H}^*$-actions on the factors.

Then $M_1 * M_2$ is not compact, but fibers over $M_1 \times M_2$ with fiber $(\mathbb{H}\setminus\{0\})/\{\pm 1\}$. Note that $\dim M_1 * M_2 = \dim M_1 + \dim M_2 + 4$. Let $r > 0$ be a real number. We may make $M_1 * M_2$ compact by dividing it by the integers, acting by dilation by e^{rn} in the fibers $(\mathbb{H}\setminus\{0\})/\{\pm 1\}$ to get a bundle over $M_1 \times M_2$ with fiber the Hopf surface. This action of the integers is given on the associated bundle $\mathscr{U}(M_1) \times \mathscr{U}(M_2)$ by multiplication by $e^{rn/2}$ in the first factor and by $e^{-rn/2}$ in the second factor.

Thus there are simple examples of compact nonsingular quaternionic manifolds involving Hopf surfaces; we exclude these because they are not simply-connected, and also because they are locally the joins of two lower-dimensional manifolds. However, both of these disadvantages may be removed by taking a nontrivial, primitive $\mathrm{U}(1)$-instanton on M_1 or M_2 or both, lifting to get a $\mathrm{U}(1)$-instanton on $M_1 * M_2$, and applying Theorem 2.2 as in Example 1. The action Ψ of $\mathrm{U}(1)$ given on the associated bundle is: $\Psi(e^{i\theta})$ acts by multiplication by $e^{r\theta/4\pi}$ on the first factor, and by $e^{-r\theta/4\pi}$ on the second.

As in Example 1, the transversality condition holds everywhere, and the result is a nonsingular, compact quaternionic manifold, which is simply-connected (because the instanton was chosen to be primitive) and fibers over $M_1 \times M_2$ with fiber the Hopf surface. It is not locally the join of two manifolds, because if it were then by simple-connectedness it would be globally so as well, and so noncompact.

As examples of suitable pairs M_1, M_2, one could take M_1 and the instanton to be any of the possibilities given in Example 1, and M_2 to be a quaternionic Kähler Riemannian symmetric space, say, or any compact self-dual 4-manifold.

4. Homogeneous hypercomplex manifolds

In the previous section it was shown that SU(3) possesses a family of homogeneous hypercomplex structures. In this section we will explore the idea of a homogeneous hypercomplex structure using the structure theory of Lie groups. We will prove that, given any compact Lie group G, there exists k with $0 \leq k \leq \max(3, \operatorname{rk} G)$ such that $\mathrm{U}(1)^k \times G$ admits a homogeneous hypercomplex structure (Theorem 4.2), and an analogous statement for homogeneous spaces.

The theory of homogeneous complex structures on compact manifolds was described in the 1950s by Wang [8] and Samelson [5], and most of what follows is a straightforward adaptation of material in those papers; the problem is to find three homogeneous complex structures satisfying the quaternionic relations.

We note that the results of Theorem 4.2 have already appeared in a Physics paper by Spindel et al. [6, p. 685, Table 1]. They approach the problem from the point of view of supersymmetry, and restrict their attention to absolutely parallelized manifolds, so that they consider only hypercomplex structures on groups and not on general homogeneous spaces. The proof we give is different from theirs, and is needed as an introduction to Theorem 4.4 and §5. We are grateful to Professor Galicki for drawing this paper to our attention.

4.1. Homogeneous hypercomplex structures on groups. The case of homogeneous hypercomplex structures on a compact group G of dimension $4n$ will be described first, for simplicity. Now in his paper [5], Samelson shows that every compact Lie group G of even dimension has a complex structure such that left translations are holomorphic mappings. This is an extension of the well-known theorem of Borel which states that the quotient of a compact Lie group by its maximal torus always has a homogeneous complex structure. We now summarize Samelson's proof.

Suppose G is a compact Lie group and H is a maximal torus of G, with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ respectively. Now as G is compact it has a finite cover G' which is the product $T \times S$ of a torus and a semisimple group. Then lifting H to $H' \subset G'$, it is clear that $H' = T \times C$, where C is a maximal torus of S. Thus for our purposes we may treat G as though it were semisimple, and H as though it were the maximal torus of a semisimple group, and perform the usual structure theory decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$.

When $\mathfrak{g}$ is a Lie algebra, denote its complexification by $\tilde{\mathfrak{g}}$. From the structure theory of Lie algebras [7, §4.3], the complexified Lie algebra $\tilde{\mathfrak{g}}$

of G is decomposed into root subspaces:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha, \tag{1}$$

where Δ is a finite subset of nonzero elements of $\tilde{\mathfrak{h}}^*$ (the *roots*), and each $\mathfrak{g}_\alpha$ is the one-dimensional subspace of $\mathfrak{g}$ defined by

$$\mathfrak{g}_\alpha = \{x : x \in \mathfrak{g}, [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}. \tag{2}$$

Samelson defines a complex structure on G by choosing a *positive system* of roots [7, p. 280], which is a set $P \subseteq \Delta$ satisfying $P \cap (-P) = \varnothing$, $P \cup (-P) = \Delta$ and α, $\beta \in P$, $\alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in P$. For let I' be a complex structure on $\mathfrak{h}$. Then if W is the set of $(1, 0)$-forms in $\tilde{\mathfrak{h}}$ with respect to I', we can define $\mathfrak{m}$ as a subset of $\tilde{\mathfrak{g}}$ by

$$\mathfrak{m} = W + \sum_{\alpha \in P} \mathfrak{g}_\alpha. \tag{3}$$

From structure theory we see that $\mathfrak{m}$ is closed under the complexified Lie bracket, and thus generates a complex subgroup M of the complexified group $\widetilde{G}$ with Lie algebra $\mathfrak{m}$. Samelson shows [5] that $\widetilde{G}/M$ is diffeomorphic to G, and as $\widetilde{G}$, M are both complex groups, this makes G a complex manifold. The complex structure on $\mathfrak{g}$ is easily described: as real vector spaces $\tilde{\mathfrak{g}} = \mathfrak{g} + \mathfrak{m}$, and this gives an identification $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{m}$, which is a quotient of complex vector spaces and so gives a complex structure on $\mathfrak{g}$. It is clear that $\mathfrak{m}$ is simply the $(1, 0)$-forms for the complex structure on $\mathfrak{g}$.

In Theorem 4.2 an analogue of this result for the hypercomplex case will be given. First we prove a preparatory lemma.

Lemma 4.1. *Let G be a compact Lie group, with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ can be decomposed as*

$$\mathfrak{g} = \mathfrak{b} + \sum_{k=1}^{n} \mathfrak{d}_k + \sum_{k=1}^{n} \mathfrak{f}_k, \tag{4}$$

where $\mathfrak{b}$ is abelian, $\mathfrak{d}_k$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$, $\mathfrak{b} + \sum_k \mathfrak{d}_k$ contains the Lie algebra of a maximal torus of G, and $\mathfrak{f}_1, \cdots, \mathfrak{f}_n$ are (possibly empty) vector subspaces of $\mathfrak{g}$, such that for each $k = 1, 2, \cdots, n$, $\mathfrak{f}_k$ satisfies the following two conditions:

(i) $[\mathfrak{d}_l, \mathfrak{f}_k] = \{0\}$ *whenever $l < k$, and*

(ii) $\mathfrak{f}_k$ *is closed under the Lie bracket with* $\mathfrak{d}_k$, *and the Lie bracket action of* $\mathfrak{d}_k$ *on* $\mathfrak{f}_k$ *is isomorphic to the sum of* m *copies of the action of* $\mathfrak{su}(2)$ *on* $\mathbb{C}^2$ *by left multiplication, for some integer* m.

Proof. Let H be a maximal torus in G with Lie algebra $\mathfrak{h}$, and Δ_1 the set of roots of $\tilde{\mathfrak{g}}$ relative to $\mathfrak{h}$. Let $\mathfrak{b}_0 = \mathfrak{g}$. Choose a highest root α_1 in Δ_1. Then the three-dimensional subspace $\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{-\alpha_1} + [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}]$ of $\tilde{\mathfrak{g}}$ is in fact a complex subalgebra of $\tilde{\mathfrak{g}}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, and its intersection with $\mathfrak{g}$ is a subalgebra isomorphic to $\mathfrak{su}(2)$. So let $\mathfrak{d}_1 = \mathfrak{g} \cap (\mathfrak{g}_{\alpha_1} + \mathfrak{g}_{-\alpha_1} + [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}])$; then $\mathfrak{d}_1$ is a subalgebra of $\mathfrak{b}_0$ isomorphic to $\mathfrak{su}(2)$.

Define $\mathfrak{b}_1$ to the centralizer of $\mathfrak{d}_1$ in $\mathfrak{b}_0$; $\mathfrak{b}_1$ is also a subalgebra. Define $\mathfrak{f}_1$ by

$$\mathfrak{f}_1 = \mathfrak{b}_0 \cap \sum_{\substack{\beta \in \Delta_1 : \\ \beta + \alpha_1 \in \Delta_1}} \mathfrak{g}_\beta + \mathfrak{g}_{\beta + \alpha_1}, \tag{5}$$

where $\mathfrak{g}_\beta$ is the root subspace for the root β. Then $\mathfrak{b}_0$ decomposes as $\mathfrak{b}_0 = \mathfrak{b}_1 + \mathfrak{d}_1 + \mathfrak{f}_1$.

This is because for every root $\beta \neq \pm\alpha_1$, $\mathfrak{g}_\beta$ appears as a summand in either $\tilde{\mathfrak{b}}_1$ or $\tilde{\mathfrak{f}}_1$, but not both, depending on whether $[\mathfrak{d}_1, \mathfrak{g}_\beta]$ is zero or nonzero respectively. Also, the $\mathfrak{g}_{\pm\alpha_1}$ appear as summands in $\tilde{\mathfrak{d}}_1$, and $\tilde{\mathfrak{h}}$ splits as $[\mathfrak{g}_{\alpha_1}, \tilde{\mathfrak{g}}_{-\alpha_1}] + (\alpha_1)^\circ$, of which the first summand comes from $\tilde{\mathfrak{d}}_1$ and the second summand from $\tilde{\mathfrak{b}}_1$. Thus from (1) it follows that $\tilde{\mathfrak{b}}_0 = \tilde{\mathfrak{b}}_1 + \tilde{\mathfrak{d}}_1 + \tilde{\mathfrak{f}}_1$, and hence the result.

Now $\mathfrak{h} \cap \mathfrak{b}_1$ is the Lie algebra of a maximal torus for the subgroup of G generated by $\mathfrak{b}_1$, and the roots Δ_2 of $\tilde{\mathfrak{b}}_1$ relative to this subalgebra are just the roots in Δ_1 that are zero on $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{-\alpha_1}]$. Either $\mathfrak{b}_1$ is abelian (and hence contained in $\mathfrak{h}$) or else we may in exactly the same way choose a highest root α_2 and decompose $\mathfrak{b}_1$ as $\mathfrak{b}_1 = \mathfrak{b}_2 + \mathfrak{d}_2 + \mathfrak{f}_2$.

By repeating this process, we obtain a succession of subalgebras $\mathfrak{b}_0, \cdots, \mathfrak{b}_n$, $\mathfrak{d}_1, \cdots, \mathfrak{d}_n$ and subspaces $\mathfrak{f}_1, \cdots, \mathfrak{f}_n$ such that $\mathfrak{b}_{i-1} = \mathfrak{b}_i + \mathfrak{d}_i + \mathfrak{f}_i$, $\mathfrak{d}_i$ is isomorphic to $\mathfrak{su}(2)$, $\mathfrak{b}_i$ is the centralizer of $\mathfrak{d}_i$ in $\mathfrak{b}_{i-1}$, and $\mathfrak{b}_n$ is abelian. Then, recalling that $\mathfrak{b}_0 = \mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{b}_n + \sum_{i=1}^{n} \mathfrak{d}_i + \sum_{i=1}^{n} \mathfrak{f}_i. \tag{6}$$

Putting $\mathfrak{b} = \mathfrak{b}_n$ gives the decomposition (4). Condition (i) is satisfied because if $l < k$ then $\mathfrak{f}_k \subset \mathfrak{b}_l$, which is the centralizer of $\mathfrak{d}_l$; thus $[\mathfrak{d}_l, \mathfrak{f}_k] = 0$.

Condition (ii) is satisfied because α_i is a highest root in $\mathfrak{b}_{i-1}$, and so by structure theory the roots of $\mathfrak{b}_{i-1}$ which do not commute with $\pm\alpha_i$ are split into pairs β, $\beta+\alpha_i$. It is then easy to see that $\mathfrak{f}_i$ splits as the sum of vector subspaces $\mathfrak{b}_{i-1}\cap(\mathfrak{g}_\beta+\mathfrak{g}_{-\beta}+\mathfrak{g}_{\beta+\alpha_i}+\mathfrak{g}_{-\beta-\alpha_i})$, and each of these is a representation of $\mathfrak{d}_i$ of the required form. q.e.d.

It will now be shown that a decomposition of this form is just what is needed to define a hypercomplex structure on $\mathrm{U}(1)^k\times G$ for some k.

Theorem 4.2 [6]. *Let G be a compact Lie group. Then there exists an integer k with $0\le k\le \mathrm{Max}(3, \mathrm{rk}\, G)$ such that $\mathrm{U}(1)^k\times G$ admits a homogeneous hypercomplex structure.*

Proof. By Lemma 4.1, the Lie algebra $\mathfrak{g}$ of G admits a decomposition

$$\mathfrak{g}=\mathfrak{b}+\sum_{k=1}^{n}\mathfrak{d}_k+\sum_{k=1}^{k}\mathfrak{f}_k \tag{7}$$

satisfying certain conditions. Now either $\dim\mathfrak{b}\le n$ or $\dim\mathfrak{b}>n$. If $\dim\mathfrak{b}\le n$, define $k=n-\dim\mathfrak{b}$, and $0\le k<\mathrm{rk}\ G$; let $m=0$. Otherwise choose $k=0, 1, 2$, or 3 such that $\dim\mathfrak{b}+k=n+4m$ for m some positive integer.

The Lie algebra of $\mathrm{U}(1)^k\times G$ is $k\mathfrak{u}(1)+\mathfrak{g}$. We will define a hypercomplex structure on this Lie algebra, which gives an almost hypercomplex structure on the group by left translation, and use Samelson's characterization of homogeneous complex structures on groups to show that the complex structures are integrable, and thus that the almost hypercomplex structure is hypercomplex.

Choose an identification (of real vector spaces) of $k\mathfrak{u}(1)+\mathfrak{b}$ with $\mathbb{H}^m+\mathbb{R}^n$; by abuse of notation we will write $k\mathfrak{u}(1)+\mathfrak{b}=\mathbb{H}^m+\mathbb{R}^n$. Note that there is a freedom in doing this of $(n+4m)^2$ parameters. In general this will mean that there are infinitely many nonisomorphic hypercomplex structures on $\mathrm{U}(1)^k\times G$. Let $(e_1, \cdots, e_n)$ be the standard basis for $\mathbb{R}^n$.

For each k, choose an isomorphism ϕ_k from $\mathfrak{su}(2)$ to $\mathfrak{d}_k$. (There are $3n$ parameters of freedom in doing this, but the different ways will lead to hypercomplex structures isomorphic up to conjugacy.)

Now the Lie algebra $\mathfrak{su}(2)$ may be written as $\langle i_1, i_2, i_3\rangle$, where i_1, i_2, and i_3 satisfy $[i_1, i_2]=2i_3$, $[i_2, i_3]=2i_1$, and $[i_3, i_1]=2i_2$. Define complex structures I_1, I_2, I_3 on $\mathfrak{g}$ by components as follows:

(a) Let the actions of I_1, I_2, I_3 on $\mathbb{H}^m$ be as usual.

(b) Let the actions of I_1, I_2, I_3 on $\mathbb{R}^n+\sum_j\mathfrak{d}_j$ be given by $I_a(e_j)=\phi_j(i_a)$, $I_a(\phi_j(i_a))=-e_j$ and $I_a(\phi_j(i_b))=\phi_j(i_c)$, $I_a(\phi_j(i_c))=-\phi_j(i_b)$ whenever (abc) is an even permutation of (123).

(c) Let the actions of I_1, I_2, I_3 on $\mathfrak{f}_j$ be given by $I_a(v) = [v, \phi_j(i_a)]$ for each $v \in \mathfrak{f}_j$.

The proof of Theorem 4.2 will be completed by the following lemma.

Lemma 4.3. *The I_1, I_2, I_3 defined above are complex structures on $k\mathfrak{u}(1) + \mathfrak{g}$ satisfying $I_1 I_2 = I_3$, and the almost complex structures on $\mathrm{U}(1)^k \times G$ generated by left translation are integrable.*

Proof. For the first part, it is clear that parts (a) and (b) lead to complex structures I_1, I_2, and I_3 satisfying $I_1 I_2 = I_3$ on their respective components, so it remains only to verify this for part (c), that is, for $\mathfrak{f}_j$. But from condition (ii) of Lemma 4.1 it can be seen that the action of $\mathfrak{d}_j$ on $\mathfrak{f}_j$ by conjugation is isomorphic to the action of $\operatorname{Im}\mathbb{H}$ on $\mathbb{H}^l$ for some l, and (c) is just a way of writing down this isomorphism.

So I_1, I_2, I_3 do form a hypercomplex structure on $k\mathfrak{u}(1) + \mathfrak{g}$. Using Samelson's results it will now be shown that they generate homogeneous complex structures by left translation.

Let a be 1, 2, or 3. Define $\mathfrak{t}$ by

$$\mathfrak{t} = \mathbb{H}^m + \mathbb{R}^n + \langle \phi_1(i_a), \cdots, \phi_n(i_a) \rangle; \tag{8}$$

then $\mathfrak{t}$ is the Lie algebra of a maximal torus T of $\mathrm{U}(1)^k \times G$. Let $V \subset k\tilde{\mathfrak{u}}(1) + \tilde{\mathfrak{g}}$ be the vector subspace of $(1,0)$-forms of I_a in $k\tilde{\mathfrak{u}}(1) + \tilde{\mathfrak{g}}$. We will construct a basis for V involving a positive system of roots for $k\tilde{\mathfrak{u}}(1)+\tilde{\mathfrak{g}}$ relative to $\mathfrak{t}$, and hence by Samelson's results show that I_a gives an integrable complex structure on $\mathrm{U}(1)^k \times G$.

We describe V by components (a), (b), (c) as above.

(a) The $(1,0)$-forms of $\mathbb{H}^m$ with respect to I_a are as usual.

(b) The $(1,0)$-forms of $\mathbb{R}^n + \sum_j \mathfrak{d}_j$ are

$$\langle e_1 + i\phi_1(i_a), \cdots, e_n + i\phi_n(i_a), \phi_1(i_b) + i\phi_1(i_c), \cdots, \phi_n(i_b) + i\phi_n(i_c) \rangle, \tag{9}$$

where (abc) is an even permutation of (123). Now $e_j + i\phi_j(i_a)$ is an element of $\tilde{\mathfrak{t}}$, and $\phi_j(i_b) + i\phi_j(i_c)$ is a root vector of $\tilde{\mathfrak{g}}$ relative to $\mathfrak{t}$. Let α_j be the root corresponding to the root vector $\phi_j(i_b) + i\phi_j(i_c)$. Also

$$\begin{aligned} [\phi_j(i_a), \phi_j(i_b) + i\phi_j(i_c)] &= -2i(\phi_j(i_b) + i\phi_j(i_c)) \\ &= \alpha_j(\phi_j(i_a))(\phi_j(i_b) + i\phi_j(i_c)), \end{aligned} \tag{10}$$

and so $\alpha_j(\phi_j(i_a)) = -2i$. Thus $\alpha_j(i\phi_j(i_a)) > 0$. (Note that $\beta(i\phi_j(i_a))$ is real for all roots β.)

(c) Now we claim that the $(1,0)$-forms of $\tilde{\mathfrak{f}}_j$ are given by

$$V \cap \tilde{\mathfrak{f}}_j = \sum_{\substack{\beta \in \Delta_j \,:\, \beta \neq \alpha_j, \\ \beta(i\phi_j(i_a)) > 0}} \mathfrak{g}_\beta, \tag{11}$$

in other words, the sum of all root subspaces of $\mathfrak{b}_{j-1}$ corresponding to roots β other than α_j that have $\beta(i\phi_j(i_a)) > 0$. Recall that in the proof of Lemma 4.1 it was shown that $\tilde{\mathfrak{f}}_j$ splits into subspaces of the form $\mathfrak{g}_\beta + \mathfrak{g}_{\beta+\alpha_j} + \mathfrak{g}_{-\beta} + \mathfrak{g}_{-\beta-\alpha_j}$, upon which the representation of $\mathfrak{d}_j$ is the complexification of the standard representation of $\mathfrak{su}(2)$ upon $\mathbb{C}^2$. It is an easy calculation to show that the $(1, 0)$-forms of this subspace are $\mathfrak{g}_{\beta+\alpha_j} + \mathfrak{g}_{-\beta}$, which verifies the claim, since by structure theory we have $\alpha_j(i\phi_j(i_a)) = -2\beta(i\phi_j(i_a))$.

Now the roots of $\tilde{\mathfrak{b}}_j$ are exactly the roots of $\tilde{\mathfrak{g}}$ that give zero when evaluated upon $\phi_1(i_a), \cdots, \phi_j(i_a)$, because these are the roots that centralize $\mathfrak{d}_1, \cdots, \mathfrak{d}_j$. So define a subset P of Δ, the set of roots of $\tilde{\mathfrak{g}}$, by

$$P = \{\alpha \in \Delta\colon \alpha(\phi_1(i_a)) = \cdots = \alpha(\phi_{j-1}(i_a)) = 0,\ \alpha(i\phi_j(i_a)) > 0 \text{ for some } j \in \{1, 2, \cdots, n\}\}. \tag{12}$$

Then P is a positive system (as defined above). But by examination we see that

$$V = V \cap \tilde{\mathfrak{t}} + \sum_{\alpha \in P} \mathfrak{g}_\alpha. \tag{13}$$

So V, the $(1, 0)$-forms of I_a, are the sum of the $(1, 0)$-forms of some complex structure on $\mathfrak{t}$ together with a positive system of roots. Therefore by [5], the left translation of I_a gives a homogeneous complex structure on $\mathrm{U}(1)^k \times G$.

4.2. General homogeneous hypercomplex manifolds. The previous section extended Samelson's result on existence of homogeneous complex structures on even-dimensional groups to the hypercomplex case. In this section we extend some of Wang's results on existence of homogeneous complex structures on general homogeneous manifolds to the hypercomplex case. His Theorem II [8, p. 15] states:

Let X be a C-subgroup of a simply-connected compact semisimple Lie group K. If K/X is even-dimensional, then K/X has a homogeneous complex structure.

Here a *C-subgroup* of K is a closed and connected subgroup whose semisimple part coincides with the semisimple part of the centralizer of a toral subgroup of K.

This theorem of Wang generalizes Samelson's result. An extension to the hypercomplex case will now be given; it will be seen that the restrictions on the subgroup X are quite severe.

First we make some definitions. Let G be a compact Lie group. We may choose a maximal torus H, and decompose $\tilde{\mathfrak{g}}$ into weights with respect to $\mathfrak{h}$. If α is any highest root, then there is a subalgebra of $\tilde{\mathfrak{g}}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ generated by $\mathfrak{g}_{\pm\alpha}$, and the intersection of this with $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$. Define a *D-subgroup* of G to be the centralizer in G of any such $\mathfrak{su}(2)$ embedded in $\mathfrak{g}$ that comes from a highest root in this way.

Now we define an *E-subgroup* of G to be any subgroup E of G such that there is a chain of subgroups and inclusions

$$G = G_0 \supset G_1 \supset \cdots \supset G_j = E, \tag{14}$$

such that G_{i+1} is a D-subgroup of G_i. We call j the *length* of E; it is well defined.

The hypercomplex version of Wang's result quoted above is:

Theorem 4.4. *Let G be a compact Lie group, and let E be an E-subgroup of G of length j. Let F be the semisimple part of E, and let X be any closed subgroup of G such that $F \subseteq X \subseteq E$. Then there exists an integer k with $0 \leq k \leq \max(3, j)$ such that $\mathrm{U}(1)^k \times G/X$ admits a homogeneous hypercomplex structure, that is, one that is preserved by left translations in $\mathrm{U}(1)^k \times G$.*

Proof. The proof is very similar to the proof of Theorem 4.2, but using where appropriate, ideas from [8] instead of ideas from [5], so it will only be briefly sketched. Using the definition of the E-subgroup E, one may carry out a decomposition of $\mathfrak{g}$ into subalgebras $\mathfrak{b}_i$, $\mathfrak{d}_i$, and subspaces $\mathfrak{f}_i$ as in Lemma 4.1, but instead of stopping when $\mathfrak{b}_n$ is abelian, we stop at $\mathfrak{b}_j = \mathfrak{e}$, where j is the length of E and $\mathfrak{e}$ is the Lie algebra of E. Now X lies between E and F, so $\mathfrak{e}$ is just $\mathfrak{x}$ plus the Lie algebra of some torus. As in the proof of Theorem 4.2, choose a suitable k and define a hypercomplex structure on $k\mathfrak{u}(1) + \mathfrak{g}/\mathfrak{x}$. Then a similar analysis to that of Lemma 4.3 shows that the complex structures I_1, I_2, I_3 give integrable complex structures when extended over the space by left translation, using methods of Wang [8].

(Note that to be able to define the left translation of the complex structure it is necessary that it should be invariant under conjugation by X. This is true because the complex structures are defined using a sequence of highest roots, and X is a subgroup of the centralizer of these roots.)

4.3. Examples. As by Theorems 4.2 and 4.4 every compact Lie group provides examples of homogeneous hypercomplex spaces, just a few interesting cases will be given. The hypercomplex structures on $\mathrm{U}(2)$ and $\mathrm{SU}(3)$ are the first examples of hypercomplex structures on the families $\mathrm{U}(2n)$ and $\mathrm{SU}(2n+1)$, and more generally on $\mathrm{U}(2k+l)/U(l)$. Also, inclusions of groups can lead to inclusions of hypercomplex manifolds;

for instance, $U(1)^n \times Sp(n)$ can appear as a hypercomplex submanifold in $U(1)^{2n} \times SO(4n)$, if the sequences of highest roots are chosen in a suitable way.

We will give $U(1) \times SO(6)$ as a worked example of Theorem 4.2, and as a worked example of Theorem 4.4 a pretty, compact, simply-connected hypercomplex 12-manifold: if $SU(2)$ is embedded in $U(3) \subset SO(6)$, it will be shown that $SO(6)/SU(2)$ is hypercomplex. Let G be $SO(6)$, and H, a maximal torus, be the diagonal matrices in $U(3) \subset SO(6)$. The Lie algebra $\mathfrak{h}$ of H is then the set of matrices in $\mathfrak{u}(3) \subset \mathfrak{so}(6)$ of the form $\mathrm{diag}(i\lambda_1, i\lambda_2, i\lambda_3)$, $\lambda_j \in \mathbb{R}$. Define coordinates (x_1, x_2, x_3) on $\tilde{\mathfrak{h}}^*$ such that (x_1, x_2, x_3) is the element of $\tilde{\mathfrak{h}}^*$ taking $\mathrm{diag}(i\lambda_1, i\lambda_2, i\lambda_3)$ to $2(x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3)$.

In these coordinates the twelve roots of $SO(6)$ are given by $(\pm i, \pm i, 0)$, $(\pm i, 0, \pm i)$, $(0, \pm i, \pm i)$. These roots are all equivalent under automorphisms of G preserving H, so every root is a highest root. Choose $(i, i, 0)$ as a highest root to generate $\mathfrak{d}_1$. This gives

$$\tilde{\mathfrak{d}}_1 = \langle \mathrm{diag}(i, i, 0)\rangle + \mathfrak{g}_{(i,i,0)} + \mathfrak{g}_{(-i,-i,0)}, \tag{15}$$

$\tilde{\mathfrak{f}}_1$ turns out to be the sum of the eight root spaces of the roots $(\pm i, 0, \pm i)$, $(0, \pm i, \pm i)$, and $\tilde{\mathfrak{b}}_1$ is

$$\tilde{\mathfrak{b}}_1 = \langle \mathrm{diag}(i, -i, 0), \mathrm{diag}(0, 0, i)\rangle + \mathfrak{g}_{(i,-i,0)} + \mathfrak{g}_{(-i,i,0)}. \tag{16}$$

There is then only one choice for $\mathfrak{d}_2$:

$$\tilde{\mathfrak{d}}_2 = \langle \mathrm{diag}(i, -i, 0)\rangle + \mathfrak{g}_{(i,-i,0)} + \mathfrak{g}_{(-i,i,0)}, \tag{17}$$

and we have $\mathfrak{f}_2 = 0$ and $\mathfrak{b}_2 = \langle \mathrm{diag}(0, 0, i)\rangle$, which is abelian. So $n = 2$, and this completes the decomposition of Lemma 4.1. To apply Theorem 4.2, we must have k such that $\dim \mathfrak{b}_n + k = n + 4m$ for some m; here $n = 2$ and $\dim \mathfrak{b}_2 = 1$, so $k = 1$ and $m = 0$ will do. Thus by Theorem 4.2, $U(1) \times SO(6)$ is hypercomplex; the freedom in the hypercomplex structure is the freedom to choose a basis (e_1, e_2) of $\mathfrak{u}(1) + \mathfrak{b}_2$, and so is of four real parameters.

To apply Theorem 4.4, we follow the decomposition of $\mathfrak{so}(6)$ above, but stop at $\mathfrak{e} = \mathfrak{b}_1$. The semisimple part of $\tilde{\mathfrak{e}}$ is

$$\tilde{\mathfrak{f}} = \langle \mathrm{diag}(i, -i, 0)\rangle + \mathfrak{g}_{(i,-i,0)} + \mathfrak{g}_{(-i,i,0)}, \tag{18}$$

and to apply Theorem 4.4 we must choose X such that $F \subseteq X \subseteq E$. Let $X = F$; then X is given by

$$X = \left\{ \begin{pmatrix} A & 0 \\ & 0 \\ 00 & 1 \end{pmatrix} : A \in SU(2) \right\} \subset U(3) \subset SO(6), \tag{19}$$

and $\mathfrak{e} = \mathfrak{x} + \langle \mathrm{diag}(0, 0, i) \rangle$ is the splitting of $\mathfrak{e}$ into $\mathfrak{x}$ and the Lie algebra of a torus. To make a hypercomplex structure we now need to take the product with $\mathrm{U}(1)^k$ for suitable k. But because the length of E is 1 and there is one dimension left over of the maximal torus, generated by $\mathrm{diag}(0, 0, i)$, we can take $k = 0$. The freedom in making the hypercomplex structure is the freedom in choosing a basis (e_1) for $\langle \mathrm{diag}(0, 0, i) \rangle$, and so is of one real parameter. So by Theorem 4.4, $G/X = \mathrm{SO}(6)/\mathrm{SU}(2)$ is a homogeneous hypercomplex manifold.

5. Homogeneous quaternionic manifolds

There is one obvious source of homogeneous quaternionic manifolds: if a quaternionic manifold has a homogeneous associated bundle, then it will be homogeneous. As the associated bundle of a quaternionic manifold is hypercomplex, we can construct homogeneous quaternionic manifolds from homogeneous hypercomplex manifolds.

In general, this sort of homogeneous quaternionic manifold will be of the form $G/\mathrm{U}(2)X$, where G/X is a compact homogeneous hypercomplex manifold, and $\mathrm{U}(2)$ embedded in G centralizes X, descends to a hypercomplex submanifold in G/X, and the action of $\mathrm{U}(2)$ on the right on G/X permutes the complex structures in the way that $\mathbb{H}^*$ does it itself by left multiplication. The problem, then, given a homogeneous hypercomplex manifold G/X as from the last section, is to find a suitable embedded (or immsered) $\mathrm{U}(2)$. Let the embedding be Φ, so that in terms of Lie algebras we seek a Lie algebra endomorphism $\Phi\colon \mathfrak{u}(2) \to \mathfrak{g}$.

This can be done using the method of construction of the last section, which involves a sequence of highest roots. Using the notation of Theorem 4.2 $\Phi(\mathfrak{su}(2))$ must be

(20)
$$\Phi(\mathfrak{su}(2)) = \langle \phi_1(i_1) + \cdots + \phi_n(i_1), \phi_1(i_2) + \cdots + \phi_n(i_2), \phi_1(i_3) + \cdots + \phi_n(i_3) \rangle .$$

This is because the hypercomplex structure is defined using $\phi_i(\mathfrak{su}(2))$, and so to permute the complex structures in the necessary way, the Lie bracket with $\Phi(\mathfrak{su}(2))$ must act on $\phi_i(\mathfrak{su}(2))$ as the Lie bracket with itself.

Recall that we said that $\Phi(\mathrm{U}(2))$ should be a hypercomplex submanifold of G/X. This will be true only if $\Phi(\mathfrak{u}(2))$ is closed under I_1, I_2, I_3. This requirement determines the fourth basis vector for $\Phi(\mathfrak{u}(2))$: it is $e_1 + \cdots + e_n$. So put

$$\begin{aligned}\Phi(\mathfrak{u}(2)) = \langle e_1 + e_n, \phi_1(i_1) + \cdots + \phi_n(i_1), \\ \phi_1(i_2) + \cdots + \phi_n(i_2), \phi_1(i_3) + \cdots + \phi_n(i_3) \rangle .\end{aligned} \tag{21}$$

Then $\Phi(\mathfrak{u}(2))$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{u}(2)$, and we can form the subgroup of G generated by it. But this subgroup may not be an embedding, or even an immersion, of $U(2)$, because it may not be closed. When $n > 1$ this is a nontrivial condition upon the hypercomplex structure chosen on G/X in §4, and is a *rationality condition*, as it simply says that the center of the embedded $U(2)$ (generated by $e_1 + \cdots e_n$) should be a closed subgroup of the maximal torus of G. The condition therefore holds for a dense subset of the homogeneous hypercomplex structures on G/X constructed in §4.

Suppose that this rationality condition holds for the choice of hypercomplex structure on G/X. Then the Lie algebra endomorphism Φ lifts to give a group homomorphism $\Phi\colon U(2) \to G$ which is an embedding or an immersion.

Proposition 5.1. *$G/\Phi(U(2))X$ is a compact, homogeneous quaternionic manifold.*

Proof. Let $U(1) \times U(1) \subset U(2)$ be the subgroup of $U(2)$ preserving the complex structure I_1 on G/X. Define $Z = G/\Phi(U(1) \times U(1))X$. Then Z is complex with complex structure I_1, as it is the quotient of G/X by $\Phi(U(1) \times U(1))$, which is a complex group with respect to I_1. Also Z fibers over $G/\Phi(U(2))X$ with fiber $U(2)/(U(1) \times U(1)) = \mathbb{CP}^1$, and it has a antiholomorphic involution σ preserving the fibers induced by $\Phi(x)$, where x is any element of $SU(2)$ that anticommutes with the $U(1) \subset SU(2)$ already fixed, and σ is antiholomorphic because acting on the right on G/X it takes I_1 to $-I_1$.

So for Z to be a twistor space for a quaternionic structure on $G/\Phi(U(2))X$, we only need to show that the normal bundle of fibers is $2a\mathcal{O}(1)$ for some integer a; by homogeneity it is enough to see this for the identity fiber $\Phi(U(2))X/X$. Let ν be the normal bundle of $\Phi(U(2))X/X$ in G/X. As $\Phi(U(2))X/X$ is a hypercomplex submanifold of G/X, which is hypercomplex, the total space of ν is hypercomplex. The left action of $\Phi(U(2))$ on ν preserves this hypercomplex structure and identities all the fibers; thus it gives a trivialization of ν.

This does not trivialize ν as a holomorphic bundle, as the flat connection on $U(2)$ it is associated with is not torsion-free. However, it can be seen that as a hypercomplex manifold, the total space of ν only depends upon a and the hypercomplex structure of $\Phi(U(2))X/X$: the Lie algebra structure in the normal directions does not affect the hypercomplex structure of ν. So the normal bundle of $\Phi(U(2))X/X$ in G/X is isomorphic as a hypercomplex manifold to a standard example.

As this standard example, let G be $GL(a+1, \mathbb{H})$. Then G acts transitively on $\mathbb{H}^{a+1}/\mathbb{Z}$, where $\mathbb{Z}$ acts by dilation; we choose the action of $\mathbb{Z}$ so that $\mathbb{H}/\mathbb{Z} \subset \mathbb{H}^{a+1}/\mathbb{Z}$ and $\Phi(\mathrm{U}(2))X/X$ are isomorphic as hypercomplex manifolds. Let X be the stabilizer of a point. Thus ν is isomorphic to the normal bundle of $\mathbb{H}/\mathbb{Z}$ in $\mathbb{H}^{a+1}/\mathbb{Z}$, so dividing by $\mathrm{U}(1) \times \mathrm{U}(1)$, the normal bundle of the identity fiber in Z is isomorphic to the normal bundle of $\mathbb{CP}^1$ in $\mathbb{CP}^{2a+1}$, which is $2a\mathscr{O}(1)$. q.e.d.

We should point out the connection between G/X and the associated bundle of $G/\Phi(\mathrm{U}(2))X$: in general G/X can be constructed from the quotient of the associated bundle of $G/\Phi(\mathrm{U}(2))X$ by a dilation action of $\mathbb{Z}$, by twisting by some homogeneous quaternionic $\mathrm{U}(1)$-connection on $G/\Phi(\mathrm{U}(2))X$, as in the first part of this paper.

5.1. An example. We consider the case of $G = \mathrm{SU}(5)$, X trivial, which is a simple example of what can happen when $n > 1$. Choose as highest weights the $\mathrm{SU}(2)$'s embedded as 2×2 matrices in the first, second and third, fourth diagonal positions of the 5×5 matrix. The construction gives that $\mathrm{SU}(5)/\Phi(\mathrm{U}(2))$ is quaternionic, where

$$\Phi(\mathrm{SU}(2)) = \left\{ \begin{pmatrix} A & \begin{matrix} 00 \\ 00 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 00 \\ 00 \end{matrix} & A & \begin{matrix} 0 \\ 0 \end{matrix} \\ 00 & 00 & 1 \end{pmatrix} : A \in \mathrm{SU}(2) \right\}, \tag{22}$$

and $\Phi(\mathrm{U}(1))$ is some closed subgroup of

$$\mathrm{U}(1) \times \mathrm{U}(1) = \left\{ \begin{pmatrix} \theta I & \begin{matrix} 00 \\ 00 \end{matrix} & \begin{matrix} 00 \\ 00 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 00 \\ 00 \end{matrix} & \eta I & \begin{matrix} 0 \\ 0 \end{matrix} & \\ 00 & 00 & \theta^{-2}\eta^{-2} & \end{pmatrix} : \theta, \eta \in \mathrm{U}(1) \right\}; \tag{23}$$

here I is the 2×2 identity matrix.

Now the important point is that for different choices of closed subgroup $\Phi(\mathrm{U}(1))$, $\mathrm{SU}(5)/\Phi(\mathrm{U}(2))$ will have different topology. This one example, then, provides us with an infinite collection of distinct, compact, simply-connected quaternionic manifolds, and in fact each of these manifolds has infinitely many distinct quaternionic structures.

5.2. Different types of homogeneity. Above we have given a way of making homogeneous quaternionic manifolds. Perhaps all compact homogeneous quaternionic manifolds with homogeneous associated bundle arise in this manner. But what about homogeneous quaternionic manifolds for which the group is not big enough to act transitively on the associated bundle? In the first part of this paper an example of this phenomenon was given: a quaternionic structure on $SU(3)$ was made with symmetry group $U(3)$, which is of too small dimension to act transitively on the associated bundle.

The quaternionic structure on such a homogeneous space G/X is given by a hypercomplex structure upon $\mathfrak{g}/\mathfrak{x}$. One approach to finding integrability conditions for a quaternionic structure specified in this way is to define, using linear functionals, first-order sections of the bundle of complex structures, and require that the Nijenhuis tensor of these sections should vanish. In this way we have shown that each of the homogeneous hypercomplex manifolds defined in §4 also admits homogeneous quaternionic structures that are not hypercomplex. The method is to construct a hypercomplex structure on the quotient of the Lie algebras using a sequence of highest roots as before, but instead of putting a standard hypercomplex structure on each of the embedded $\mathfrak{u}(2)$'s to choose a hypercomplex structure corresponding to a quaternionic structure on $U(2)$ that is not hypercomplex. The calculation mentioned above then shows that the almost quaternionic structure defined by translation is quaternionic, but not hypercomplex.

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